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A Characterization of Descartes Systems in Haar Subspaces

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A characterization of existence of Descartes systems in Haar subspaces is given. Moreover, it is shown that the functions in such systems can be represented as products of piecewise strictly monotone functions.

INTRODUCTION

Let M be a subset of $\mathbb{R}$ which contains at least n points and let $F(M) = \{f: M \to \mathbb{R}\}$. Moreover, let U denote an n-dimensional subspace of $F(M)$. A *Descartes basis* in U is a basis $\{u_1, ..., u_n\}$ of U such that, for any $1 \leqslant i_1 < \cdots < i_m \leqslant n$ and any points $t_1 < \cdots < t_m$ in M,

$$D\begin{pmatrix} u_{i_1} \cdots u_{i_m} \\ t_1 \cdots t_m \end{pmatrix} = \det\,(u_{i_j}(t_k))_{j=1}^{m}\,{}_{k=1}^{m} \neq 0,$$

$1 \leqslant m \leqslant n$. Such a basis $\{u_1, ..., u_n\}$ is also called a *Descartes system* in U on M. A system $\{u_1, ..., u_n\}$ in U is called a *Markoff system* if, for any points $t_1 < \cdots < t_m$ in M,

$$D\begin{pmatrix} u_1 \cdots u_m \\ t_1 \cdots t_m \end{pmatrix} \neq 0,$$

$1 \leqslant m \leqslant n$. Moreover, a system $\{u_1, ..., u_n\}$ in U is called a *sign-regular*

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Descartes system if, for any $1 \leqslant i_1 < \cdots < i_m \leqslant n$, there exists an $\varepsilon \in \{-1, 1\}$ such that, for any $t_1 < \cdots < t_m$ in M,

$$\varepsilon D\begin{pmatrix} u_{i_1} \cdots u_{i_m} \\ t_1 \cdots t_m \end{pmatrix} > 0,$$

$1 \leqslant m \leqslant n$. Analogously we define *sign-regular Markoff systems* in U. A Markoff system $\{u_1, ..., u_n\}$ in U is called *normed* if $u_1 \equiv 1$ on M. (Obviously, if M is an interval, then every Descartes (resp. Markoff) system is a sign-regular Descartes (resp. Markoff) system.)

It is well known that U is called a *Haar subspace* of $F(M)$, if there exists a basis $\{u_1, ..., u_n\}$ in U such that for any points $t_1 < \cdots < t_n$ in M,

$$D\begin{pmatrix} u_1 \cdots u_n \\ t_1 \cdots t_n \end{pmatrix} \neq 0.$$

In the following we are interested in such Haar spaces which admit Descartes systems. We give a sufficient condition ensuring the existence of Descartes systems. Under some weak additional hypotheses we are able to prove the more difficult converse result. In particular it follows that if M is a closed interval, then there exists a Descartes system in a subspace U of $C(M)$ if and only if for every interval $\tilde{M} \supset M$ there exists a Haar subspace $\tilde{U}$ of $C(\tilde{M})$ such that $\tilde{U}|_M = U$. We give an example of a Haar subspace U of $C(M)$ where $M = [a, b]$ such that U does not admit a Descartes system on (a, b) which implies that U cannot be extended to a Haar space on $(a-d, b)$ or on $(a, b+d)$ for any $d > 0$. Moreover, we show that the functions in a Descartes system can be represented as products of piecewise strictly monotone functions. Finally, from the above results we derive a characterization of those normed sign-regular Markoff systems which admit sign-regular Descartes systems using integral representations.

Independently and simultaneously Zalik and Zwick [5] have also studied the problem of existence of Descartes systems in Haar spaces and have obtained the statement of Corollary 2.4 and a statement similar to Theorem 2.2.

1. Representation of Descartes Systems

In this section we give a representation of Descartes systems.

THEOREM 1.1. *Let $\{u_1, ..., u_n\} \subset F(M)$ be a Descartes system. Then*

(1) *there exist functions $w_i \in F(M)$, $1 \leqslant i \leqslant n-1$, such that*

$$u_{i+1} = u_1 w_1 \cdots w_i, \qquad 1 \leqslant i \leqslant n-1,$$

where w_i is strictly monotone on every connected component of M and $w_i(x) \neq 0$ for every $x \in M$, $1 \leqslant i \leqslant n-1$;

(2) *if u_i has constant sign on M, $1 \leqslant i \leqslant n$, and if for each $1 \leqslant i \leqslant n-1$, $D\binom{u_i\ u_{i+1}}{t_1\ t_2}$ has constant sign for any points $t_1 < t_2$ in M, the functions w_i are even strictly monotone on M, $1 \leqslant i \leqslant n-1$;*

(3) *if $u_i > 0$ on M, $1 \leqslant i \leqslant n$, and if for all $i, j \in \{1, ..., n\}$, $i \neq j$, $D\binom{u_i\ u_j}{t_1\ t_2}$ has constant sign for any points $t_1 < t_2$ in M, the system $\{u_1, ..., u_n\}$ can be rearranged to the system $\{u_{l_1}, ..., u_{l_n}\}$ such that there exist strictly increasing functions $\tilde{w}_i \in F(M)$, $1 \leqslant i \leqslant n-1$ with*

$$u_{l_{j+1}} = u_{l_1} \tilde{w}_1 \cdots \tilde{w}_j, \qquad 1 \leqslant j \leqslant n-1;$$

(4) *if $u_i \in C(M)$, $1 \leqslant i \leqslant n$, then $w_i, \tilde{w}_i \in C(M)$, $1 \leqslant i \leqslant n-1$.*

Proof. (1) Since by hypothesis span $\{u_1, u_2\}$ is a Haar subspace of $F(M)$, and $u_i(x) \neq 0$ for every $x \in M$, $i = 1, 2$, span $\{1, u_2/u_1\}$ is also a Haar space on M. This implies that u_2/u_1 is strictly monotone on every connected component of M. Hence $u_2 = u_1 w_1$ where w_1 has the desired property.

Repeated application of this argument to the Haar spaces span $\{u_2, u_3\}, ...,$ span $\{u_{n-1}, u_n\}$ yields functions $w_i = u_{i+1}/u_i$, $2 \leqslant i \leqslant n-1$, such that w_i is strictly monotone on every connected component of M. Hence

$$u_{i+1} = u_i w_i = \cdots = u_1 w_1 \cdot \cdots \cdot w_i, \qquad 1 \leqslant i \leqslant n-1.$$

(2) Assume that w_1 is not strictly monotone on M. We only consider the case when $w_1(t_1) < w_1(t_2) > w_1(t_3)$ for some points $t_1 < t_2 < t_3$ in M. Then the proof of (1) implies that $u_2(t_1)/u_1(t_1) < u_2(t_2)/u_1(t_2) > u_2(t_3)/u_1(t_3)$.

Since u_1 and u_2 have constant sign on M, it follows that

$$D\begin{pmatrix} u_1 & u_2 \\ t_1 & t_2 \end{pmatrix} \cdot D\begin{pmatrix} u_1 & u_2 \\ t_2 & t_3 \end{pmatrix} < 0,$$

a contradiction.

Using the proof of (1) and the above arguments we can show that also the function w_i is strictly monotone on M for $2 \leqslant i \leqslant n-1$.

(3) Since by hypothesis span $\{u_i, u_j\}$ is a Haar subspace of $F(M)$ for any $i, j \in \{1, ..., n\}$, $i \neq j$, and u_i has constant sign on M, $1 \leqslant i \leqslant n$, it follows from (1), (2) that u_i/u_j is strictly monotone on M.

If u_{i+1}/u_i is strictly increasing on M for every $1 \leqslant i \leqslant n-1$, then setting $\tilde{w}_i = u_{i+1}/u_i$ we obtain

$$u_{i+1} = u_1 \tilde{w}_1 \cdot \cdots \cdot \tilde{w}_i, \qquad 1 \leqslant i \leqslant n-1,$$

and the statement follows.

In the other case we rearrange the system $\{u_1, \ldots, u_n\}$ as follows: Assume that the rearrangement $\{u_{i_1}, u_{i_2}, \ldots, u_{i_n}\}$ is given. Moreover assume that for some integer $i_j \in \{1, \ldots, n-1\}$, $u_{i_{j+1}}/u_{i_j}$ is strictly decreasing on M. Then we define a new arrangement by

$$\{u_{i_1}, \ldots, u_{i_{j-1}}, u_{i_{j+1}}, u_{i_j}, u_{i_{j+2}}, \ldots, u_{i_n}\}.$$

It is easily verified that the same arrangement cannot occur twice. Therefore, since there are only a finite number of distinct arrangements, we arrive after a finite number of steps at a system $\{u_{l_1}, \ldots, u_{l_n}\}$ such that $u_{l_{j+1}}/u_{l_j}$ is strictly increasing on M, $1 \leqslant j \leqslant n-1$.

Then setting $\tilde{w}_j = u_{l_{j+1}}/u_{l_j}$ we obtain

$$u_{l_{j+1}} = u_{l_1} \tilde{w}_1 \cdot \cdots \cdot \tilde{w}_j, \qquad 1 \leqslant j \leqslant n-1.$$

(4) The statement follows directly from the proof of (1)–(3).

COROLLARY 1.2. *If $\{u_1, \ldots, u_n\}$ is a Descartes system in $C[a, b]$ where $[a, b]$ is a real compact interval, then there exist functions $w_i \in C[a, b]$, $1 \leqslant i \leqslant n-1$, which are strictly monotone on $[a, b]$ such that*

$$u_{i+1} = u_1 w_1 \cdot \cdots \cdot w_i, \qquad 1 \leqslant i \leqslant n-1.$$

If in particular $u_i > 0$ on $[a, b]$, $1 \leqslant i \leqslant n$, then $\{u_1, \ldots, u_n\}$ can be rearranged to a system $\{u_{l_1}, \ldots, u_{l_n}\}$ such that

$$u_{l_{j+1}} = u_{l_1} \tilde{w}_1 \cdot \cdots \cdot \tilde{w}_j, \qquad 1 \leqslant j \leqslant n-1,$$

where $\tilde{w}_i$ is a strictly increasing function in $C[a, b]$, $1 \leqslant i \leqslant n-1$.

The converse of Theorem 1.1 is not true for $n \geqslant 3$ in general as the following example shows.

EXAMPLE 1.3. Let the functions $\{u_1, u_2, u_3\} \subset C[0, \pi]$ be defined by $u_1(x) = 1$, $u_2(x) = w_1(x)$, $u_3(x) = w_1(x)\, w_2(x)$ for every $x \in [0, \pi]$ where $w_1(x) = 1/(\cos x + 2)$ and $w_2(x) = x + 1$. Obviously, w_1 and w_2 are strictly increasing and positive on $[0, \pi]$.

But span $\{u_1, u_2, u_3\}$ is not a Haar space on $[0, \pi]$, since the function $u(x) = (\cos x + (2/\pi)\, x - 1)/(\cos x + 2)$ has the zeros $0, \pi/2, \pi$.

2. MAIN RESULTS

We begin by giving a sufficient condition ensuring the existence of Descartes systems.

THEOREM 2.1. *Let U denote an n-dimensional Haar subspace of $F(M)$. Assume that there exist distinct points $z_1, ..., z_n$ in $\mathbb{R}\backslash M$ such that there exists an n-dimensional Haar subspace $\tilde{U}$ of $F(\tilde{M})$ where $\tilde{M} = M \cup \{z_1, ..., z_n\}$ satisfying $\tilde{U}|_M = U$. Then there exists a Descartes system $\{u_1, ..., u_n\}$ in U on M.*

Proof. Since $\tilde{U}$ is a Haar space on $\tilde{M}$, there exist (unique) functions $\tilde{u}_i \in \tilde{U}$ defined by

$$\tilde{u}_i(z_j) = \delta_{ij}, \qquad 1 \leqslant i, j \leqslant n.$$

Set $u_i = \tilde{u}_i|_M$, $1 \leqslant i \leqslant n$, and let $\{u_{i_1}, ..., u_{i_m}\}$ be a subsystem of $\{u_1, ..., u_n\}$. Then $u_{i_j}(z_r) = 0$ for every $r \in \{1, ..., n\} \backslash \{i_1, ..., i_m\}$. Hence every nontrivial function $u \in \operatorname{span}\{u_{i_1}, ..., u_{i_m}\}$ has at most $m-1$ zeros in M. This implies that $\operatorname{span}\{u_{i_1}, ..., u_{i_m}\}$ is a Haar subspace of $F(M)$.

Thus we have shown that $\{u_1, ..., u_n\}$ is a Descartes system in U on M.

Remark. Let U denote an n-dimensional Haar subspace of $C[a, b]$ where $[a, b]$ is a compact real interval and let $0 < \varepsilon < b - a$ be given. Then $\tilde{U} = U|_{[a+\varepsilon, b]}$ contains a sign-regular Descartes system.

Under some weak additional hypotheses we now prove the more difficult converse of Theorem 2.1.

THEOREM 2.2. *Let* $\inf M \notin M$, $\sup M \notin M$, $\inf M \in \mathbb{R}$, *and for any points $x, y \in M$ with $x < y$ there exists a point $z \in M$ with $x < z < y$. Set $\tilde{M} = \{\inf M\} \cup M$. Assume that U is an n-dimensional subspace of $F(\tilde{M})$ which contains a sign-regular Descartes system $\{u_1, ..., u_n\}$ on $\tilde{M}$. Then for every $d > 0$ there exists a space $U_d = \operatorname{span}\{\tilde{u}_1, ..., \tilde{u}_n\}$ on* $(\inf M - d, \inf M) \cup \tilde{M}$ *such that*

$$U_d|_{\tilde{M}} = U; \tag{2.1}$$

every function $u \in U_d$ is continuous on $(\inf M - d, \inf M]$; (2.2)

for any points $t_1 < \cdots < t_n$ in $(\inf M - d, \inf M) \cup \tilde{M}$,

$$\varepsilon D\begin{pmatrix} \tilde{u}_1 \cdots \tilde{u}_n \\ t_1 \cdots t_n \end{pmatrix} > 0,$$

where $\varepsilon \in \{-1, 1\}$; i.e. U_d is a Haar space. (2.3)

If in particular $1 \in U$, then there exists a normed sign-regular Markoff system in U_d. (2.4)

The proof of the above theorem is based on the following result.

LEMMA 2.3. *Let M and $\tilde{M}$ be defined as in Theorem 2.2. Moreover, let U denote an n-dimensional subspace of $F(\tilde{M})$ such that there exist $\lim_{t \to \sup M, t \in M} u(t)$ for every $u \in U$ and a system $\{u_1, ..., u_n\}$ in U with $u_1 \equiv 1$ on $\tilde{M}$ and*

$$\varepsilon D\begin{pmatrix} u_1 u_{i_1} \cdots u_{i_m} \\ t_1 t_2 \cdots t_{m+1} \end{pmatrix} > 0$$

for any $2 \leqslant i_1 < \cdots < i_m \leqslant n$ and any points $t_1 < \cdots < t_{m+1}$ in $\tilde{M}$ where $\varepsilon = \varepsilon(i_1, ..., i_m) \in \{-1, 1\}$, $1 \leqslant m \leqslant n-1$. Then there exists a sign-regular Descartes system $\{v_1, ..., v_n\}$ in U on $\tilde{M}$.

Proof. Define a system $\{v_1, ..., v_n\}$ in U by

$$v_1 \equiv 1$$
$$v_i = u_i - \Big(\lim_{\substack{t \to \sup M \\ t \in M}} u_i(t)\Big), \qquad 2 \leqslant i \leqslant n.$$

Since by assumption $\lim_{t \to \sup M, t \in M} u_i(t)$ is a real number, every v_i is well defined on $\tilde{M}$, $2 \leqslant i \leqslant n$.

We show that this system has the desired property. To do this let $\{v_{j_1}, ..., v_{j_m}\}$ be a subsystem where $1 \leqslant j_1 < \cdots < j_m \leqslant n$. If $j_1 = 1$, then span $\{v_{j_1}, ..., v_{j_m}\} = \text{span}\ \{u_1, u_{j_2}, ..., u_{j_m}\}$ and the statement follows immediately.

Therefore let $j_1 > 1$. Assume that the statement is false. Then by Lemma 3.1 in [2] there exists a function $\tilde{v} \in \text{span}\{v_{j_1}, ..., v_{j_m}\}$ and points $x_1 < \cdots < x_{m+1}$ in $\tilde{M}$ satisfying

$$(-1)^i\, \tilde{v}(x_i) \geqslant 0, \qquad 1 \leqslant i \leqslant m+1.$$

Since $j_1 > 1$, it follows from the definition of $v_2, ..., v_n$ that $\lim_{t \to \sup M, t \in M} \tilde{v}(t) = 0$. Therefore and by the properties of M there exist points $y_1 < \cdots < y_{m+2}$ in $\tilde{M}$ such that

$$(-1)^i\,(\tilde{v}(y_{i+1}) - \tilde{v}(y_i)) \geqslant 0, \qquad 1 \leqslant i \leqslant m+1.$$

This contradicts Theorem 8.8 in [2], because $\tilde{v} \in \text{span}\ \{u_1, u_{j_1}, ..., u_{j_m}\}$ and by hypothesis this system is a normed sign-regular Markoff system on $\tilde{M}$.

Proof of Theorem 2.2. We may assume that $u_i > 0$ on $\tilde{M}$, $1 \leqslant i \leqslant n$. Let $d > 0$ be given. We proceed by induction on n. For $n = 1$ the statement is easily verified. Assuming the result holds for $n-1$, we now prove it for n.

At first it follows from Theorem 1.1 that there exists a rearrangement $\{u_{l_1}, ..., u_{l_n}\}$ of $\{u_1, ..., u_n\}$ such that

$$u_{l_{j+1}} = u_{l_1} w_1 \cdot \cdots \cdot w_j, \qquad 1 \leqslant j \leqslant n-1,$$

where w_i is strictly increasing and positive on $\tilde{M}$, $1 \leqslant i \leqslant n-1$. Without loss of generality we may assume that $l_j = j$, $1 \leqslant j \leqslant n$, and $u_1 \equiv 1$. Then

$$u_{j+1} = w_1 \cdot \cdots \cdot w_j, \qquad 1 \leqslant j \leqslant n-1.$$

Since u_n is strictly increasing on $\tilde{M}$, there exists the function u_n^{-1} and therefore the functions

$$v_1 = u_1 \circ u_n^{-1},$$
$$v_{j+1} = u_{j+1} \circ u_n^{-1} = \tilde{w}_1 \cdot \cdots \cdot \tilde{w}_j, \qquad 1 \leqslant j \leqslant n-1,$$

where $\tilde{w}_i = w_i \circ u_n^{-1}$, $1 \leqslant i \leqslant n-1$. Then obviously $v_i \in F(\tilde{D})$ where $\tilde{D} = u_n(\tilde{M})$, $1 \leqslant i \leqslant n$, and $\tilde{w}_i$ is strictly increasing and positive on $\tilde{D}$, $1 \leqslant i \leqslant n-1$. Moreover it follows that $v_1 \equiv 1$ and $v_n(x) = x$ for every $x \in \tilde{D}$.

Set $D = u_n(M)$, $a = \inf D$, $b = \sup D$. It is easily verified (see Lemma 14.2 in [2]) that $\inf D \notin D$, $\sup D \notin D$, $\inf D \in \mathbb{R}$, and for any points $x, y \in \tilde{D}$ with $x < y$ there exists a point $z \in \tilde{D}$ with $x < z < y$. Moreover Lemma 14.2 in [2] implies that $\{v_1, ..., v_n\}$ is a sign-regular Descartes system on $\tilde{D}$.

Let $V = \text{span}\,\{v_1, ..., v_n\}$. Then the space V and the set $\tilde{D}$ have the same properties as the space U and the set $\tilde{M}$. In addition, V contains the functions $v_1 \equiv 1$ and $v_n(x) = x$ $(x \in \tilde{D})$ which will be necessary for our later considerations.

Now using Theorem 11.3 in [2] there exists, for any $v \in V$ and any $x \in D$,

$$D_+ v(x) = \lim_{\substack{t \to x^+ \\ t \in D}} \frac{v(t) - v(x)}{v_n(t) - v_n(x)}.$$

This implies that $D_+ v_1 \equiv 0$ and $D_+ v_n \equiv 1$. Moreover, since $v_2, ..., v_n$ and $\tilde{w}_1, ..., \tilde{w}_{n-1}$ are strictly increasing and positive on $\tilde{D}$, for any $x \in D$ and $2 \leqslant i \leqslant n$,

$$0 \leqslant D_+ v_i(x) = \lim_{\substack{t \to x^+ \\ t \in D}} \frac{(\tilde{w}_1 \cdot \cdots \cdot \tilde{w}_{i-1})(t) - (\tilde{w}_1 \cdot \cdots \cdot \tilde{w}_{i-1})(x)}{(\tilde{w}_1 \cdot \cdots \cdot \tilde{w}_{n-1})(t) - (\tilde{w}_1 \cdot \cdots \cdot \tilde{w}_{n-1})(x)}$$

$$\leqslant \frac{1}{(\tilde{w}_i \cdot \cdots \cdot \tilde{w}_{n-1})(x)} \leqslant \frac{1}{(\tilde{w}_1 \cdot \cdots \cdot \tilde{w}_{n-1})(a)} = K,$$

where $K \in \mathbb{R}$.

Since by hypothesis $\{v_1, ..., v_n\}$ is a sign-regular Descartes system on $\tilde{D}$ and $v_1 \equiv 1$ on $\tilde{D}$, it follows from Theorem 11.3 in [2] that

$$\varepsilon D \begin{pmatrix} D_+ v_n & D_+ v_{j_1} & \cdots D_+ v_{j_m} \\ \tilde{t}_1 & \tilde{t}_2 & \cdots \tilde{t}_{m+1} \end{pmatrix} > 0 \tag{2.5}$$

for any $2 \leqslant j_1 < \cdots < j_m \leqslant n-1$ and any points $\tilde{t}_1 < \cdots < \tilde{t}_{m+1}$ in D where $\varepsilon = \varepsilon(j_1, ..., j_m) \in \{-1, 1\}$, $0 \leqslant m \leqslant n-2$. In particular, $D_+ v_i$ is strictly monotone on D, $2 \leqslant i \leqslant n-1$. Hence there exist $\lim_{x \to a, x \in D} D_+ v_i(x)$ and $\lim_{x \to b, x \in D} D_+ v_i(x)$, $2 \leqslant i \leqslant n-1$. Set $D_+ v_i(a) = \lim_{x \to a, x \in D} D_+ v_i(x)$, $2 \leqslant i \leqslant n-1$, and $D_+ V = \text{span}\{D_+ v_n, D_+ v_2, ..., D_+ v_{n-1}\}$. Then by Lemma 12.1 and Lemma 12.2 in [2] and the above definition of $D_+ v_i(a)$, $D_+ V \subset C_+(\tilde{D}) = \{f \in F(\tilde{D}) : f \text{ is continuous from the right}\}$.

Then as in the proof of Proposition 2.7 we can show that

$$\tilde{\varepsilon} D \begin{pmatrix} D_+ v_n & D_+ v_{j_1} & \cdots & D_+ v_{j_m} \\ \tilde{t}_1 & \tilde{t}_2 & \cdots & \tilde{t}_{m+1} \end{pmatrix} > 0$$

for any $2 \leqslant j_1 < \cdots < j_m \leqslant n-1$ and any points $\tilde{t}_1 < \cdots < \tilde{t}_{m+1}$ in $\tilde{D}$ where $\tilde{\varepsilon} = \tilde{\varepsilon}(j_1, ..., j_m) \in \{-1, 1\}$.

Now by Lemma 2.3 there exists a sign-regular Descartes system $\{h_1, ..., h_{n-1}\}$ in $D_+ V$ on $\tilde{D}$ where $h_1 \equiv 1$ on $\tilde{D}$.

Now let $\tilde{d} > 0$ be given. Then by the induction hypothesis there exists a space $D_+ V_{\tilde{d}} = \text{span}\{\tilde{h}_1, ..., \tilde{h}_{n-1}\}$ on $(a - \tilde{d}, a) \cup \tilde{D}$ satisfying (2.1)–(2.3). Since $1 \in D_+ V$, by the induction hypothesis we may assume that $\{\tilde{h}_1, ..., \tilde{h}_{n-1}\}$ is a normed sign-regular Markoff system in $D_+ V_{\tilde{d}}$. By Lemma 14.3 in [2] there exists an n-dimensional weak Chebyshev subspace $\bar{V}$ of $C(I)$ where $I = (a, b)$ (i.e., if $(e_1, ..., e_n)$ is a basis of $\bar{V}$, then for any points $t_1 < \cdots < t_n$ in (a, b), $\varepsilon D\binom{e_1 \cdots e_n}{t_1 \cdots t_n} \geqslant 0$ where $\varepsilon \in \{-1, 1\}$) such that

$$\bar{V}|_D = V; \tag{2.6}$$

for every $\bar{v} \in \bar{V}$ and every $x \in I$, there exists

$$D_+ \bar{v}(x) = \lim_{t \to x^+} \frac{\bar{v}(t) - \bar{v}(x)}{t - x}; \tag{2.7}$$

$D_+ \bar{V}$ is an $(n-1)$-dimensional weak Chebyshev space with $D_+ \bar{V}|_D = D_+ V$; (2.8)

$$D_+ \bar{v} \in C_+(I) \qquad \text{for every } \bar{v} \in \bar{V}. \tag{2.9}$$

Hence there exists a basis $\{\bar{h}_1, ..., \bar{h}_{n-1}\}$ of $D_+ \bar{V}$ such that $\bar{h}_i|_D = \tilde{h}_i$, $1 \leqslant i \leqslant n-1$. We set $D_+ \bar{V}_{\tilde{d}} = \text{span}\{g_1, ..., g_{n-1}\}$ where

$$g_i(x) = \begin{cases} \tilde{h}_i(x) & \text{if } x \in (a - \tilde{d}, a] \\ \bar{h}_i(x) & \text{if } x \in I, \end{cases}$$

$1 \leqslant i \leqslant n-1$. This implies that $D_+ \bar{V}_{\tilde{d}}|_I = D_+ \bar{V}$ and $D_+ \bar{V}_{\tilde{d}} \subset C_+(\tilde{I})$ where $\tilde{I} = (a - \tilde{d}, b)$. Since $\{\tilde{h}_1, ..., \tilde{h}_{n-1}\}$ is a normed sign-regular Markoff system in $D_+ V_{\tilde{d}}$, it is obvious that span $\{g_1, ..., g_i\}$ is an i-dimensional weak

Chebyshev space, $1 \leqslant i \leqslant n-1$. Otherwise for some $i \in \{1, ..., n-1\}$ there would exist a function $\tilde{g} \in \text{span}\{g_1, ..., g_i\}$ and points $x_1 < \cdots < x_i$ in $\tilde{I}$ with $(-1)^j \tilde{g}(x_j) > 0$, $1 \leqslant j \leqslant i$. Then using the construction of $\bar{V}$ in the proof of Lemma 14.3 in [2] we obtain a function $\hat{g} \in \text{span}\{\tilde{h}_1, ..., \tilde{h}_i\}$ and i points $y_1 < \cdots < y_i$ in $(a-\tilde{d}, a) \cup \tilde{D}$ with $(-1)^j \hat{g}(y_j) > 0$, $1 \leqslant j \leqslant i$, a contradiction. Now for every $i \in \{1, ..., n-1\}$ we define

$$W_i = \left\{ w \in C(\tilde{I}) \colon w(x) = \int_c^x g(t)\,dt + \alpha,\, g \in \text{span}\{g_1, ..., g_i\},\, c \in \tilde{I},\, x \in \tilde{I} \right\}.$$

Since every $g \in C_+(\tilde{I})$, the set W_i is well defined. (This was the reason for transforming the given space U to the space V.) Moreover, it follows from Lemma 13.2 in [2] that W_i is an $(i+1)$-dimensional weak Chebyshev subspace of $C(\tilde{I})$.

Now we show that there exists a basis $\{f_0, ..., f_i\}$ of W_i, $1 \leqslant i \leqslant n-1$, such that $f_0 \equiv 1$ on $\tilde{I}$ and

$$\varepsilon D \begin{pmatrix} f_0 \cdots f_i \\ t_0 \cdots t_i \end{pmatrix} > 0 \tag{2.10}$$

for any points $t_0 < \cdots < t_i$ in $\hat{D}$ where $\hat{D} = (a-\tilde{d}, a) \cup \tilde{D}$ and $\varepsilon \in \{-1, 1\}$. Assume that there exist points $t_0 < \cdots < t_{i+1}$ in $\hat{D}$ and a function $f \in W_i \setminus \{0\}$ such that

$$(-1)^j (f(t_{j+1}) - f(t_j)) \geqslant 0, \qquad 0 \leqslant j \leqslant i.$$

By definition of W_i there is a function $g \in \text{span}\{g_1, ..., g_i\}$ with

$$(-1)^j \int_{t_j}^{t_{j+1}} g(t)\,dt \geqslant 0, \qquad 0 \leqslant j \leqslant i.$$

Assume that for some j, $(-1)^j g(t) < 0$ for every $t \in \hat{D} \cap (t_j, t_{j+1})$. Since by construction of $\bar{V}$ (see Lemma 14.3 in [2]) every $g \in D_+ \bar{V}_{\tilde{d}}$ is piecewise constant on $\tilde{I} \setminus \hat{D}$, it follows that $(-1)^j g(t) < 0$ for every $t \in (t_j, t_{j+1})$. Then $g \in C_+(\tilde{I})$ implies that

$$(-1)^j \int_{t_j}^{t_{j+1}} g(t)\,dt < 0, \text{ a contradiction.}$$

Thus we have shown that for every $j \in \{0, ..., i\}$ there exists a point $z_j \in (t_j, t_{j+1}) \cap \hat{D}$ such that $(-1)^j g(z_j) \geqslant 0$. However, this contradicts Lemma 3.1 in [2]. Then (2.10) follows from Lemma 8.2 in [2].

It follows from Lemma 14.5 in [2] that $W|_{\tilde{D}} = V$ where $W = W_{n-1}$. Now using the extension W of V we can easily obtain the desired extension of U. To do this define $\tilde{u}_n(x) = u_n(x)$ for every $x \in \tilde{M}$, $\tilde{u}_n(\inf M - d) = a - \tilde{d}$, $\tilde{u}_n$ continuous and strictly increasing on $[\inf M - d, \inf M]$. Set

$U_d = \text{span}\{f_0 \circ \tilde{u}_n, ..., f_{n-1} \circ \tilde{u}_n\}$. Then by the construction of V and by Lemma 14.2 in [2] U_d has the desired properties (2.1)–(2.3). This proves Theorem 2.2.

Remark. Under the assumptions $\sup M \in \mathbb{R}$ and $\tilde{M} = \{\sup M\} \cup M$, the space U can be extended to a subspace U_d defined on $\tilde{M} \cup (\sup M, \sup M + d)$ satisfying the properties (2.1)–(2.3) $(d > 0)$.

For the most important case when M is an interval the following equivalent statements are an immediate consequence of Theorem 2.1 and Theorem 2.2.

COROLLARY 2.4. *Let $M = [a, b)$, a real subinterval, and let U denote an n-dimensional subspace of $C(M)$. Then the following conditions are equivalent:*

There exists a Descartes system in U on M; (2.11)

For every $d > 0$ there exists an n-dimensional Haar subspace U_d of $C((a-d, b))$ such that $U_d|_M = U$. (2.12)

Analogously the Haar space U can be extended on the subinterval $(a, b+d)$ (resp. on $[a, b+d)$), if $b \in M$ (for every $d > 0$) and on the subinterval $(a-d, b+\tilde{d})$, if $M = [a, b]$ (for every $d > 0$ and every $\tilde{d} > 0$).

It follows directly from the definitions that every Descartes system $\{u_1, ..., u_n\}$ is a Markoff system and that $U = \text{span}\{u_1, ..., u_n\}$ is a Haar space. Now we show that the converse is not true in general. Let $[a, b]$ be a real subinterval and let U denote an n-dimensional Haar space on (a, b). Then by Theorem 7.7 in [2] there exists a Markoff system in U on (a, b). However, if U is a Haar space on $[a, b)$ or on $(a, b]$ or on $[a, b]$, then the above result is not true in general (see Sect. 10 in [2]). In particular this implies that there exist Haar subspaces U which do not admit Descartes systems.

Although every Haar space U on $[a, b]$ contains a Markoff system on (a, b), U does not admit a Descartes system on (a, b) in general. Then Corollary 2.4 implies that U cannot be extended to a Haar space on $(a-d, b]$ or on $[a, b+d)$ for any $d > 0$.

EXAMPLE 2.5. Let $M = [-1, 1]$ and let $U = \text{span}\{u_1, u_2, u_3\}$ where $u_1(x) = 1$, $u_2(x) = x(1-x)$, and $u_3(x) = (1-x^2)(1-x)$ for every $x \in [-1, 1]$.

CLAIM. *U has no Descartes basis on $(-1, 1)$.*

Proof. In [2, Sect. 10] it was verified that U is a Haar space on $[-1, 1]$. Moreover it was shown that U does not contain a two-dimensional Haar subspace. Hence U has no Descartes basis on $[-1, 1]$.

Now assume that there exists a Descartes basis $\{v_1, v_2, v_3\}$ of U on $(-1, 1)$. At first we show that $v_i(x) \neq 0$ for every $x \in [-1, 1]$ and some $i \in \{1, 2, 3\}$. Suppose that for $i = 1, 2, 3$, $v_i(-1) = 0$ or $v_i(1) = 0$. Since $v_i(-1) = 0$ for every $i \in \{1, 2, 3\}$ contradicts the Haar property of U on $[-1, 1]$, we may assume that $v_1(-1) \neq 0$ and $v_1(1) = 0$. Since span $\{v_1\}$ is a Haar space on $(-1, 1)$, it follows that $v_1(x) \neq 0$ for every $x \in (-1, 1)$. Hence $v_1 = cu_3$ where c is a nonzero real number. This implies that $v_1(-1) = 0$, a contradiction.

Thus we have shown that for some $i \in \{1, 2, 3\}$, $v_i(x) \neq 0$ for every $x \in [-1, 1]$. Without loss of generality we may assume that $v_1(x) \neq 0$ for every $x \in [-1, 1]$. Since $\{v_1, v_2, v_3\}$ is a Descartes basis of U on $(-1, 1)$, it follows that span $\{v_1, v_2\}$ satisfies the Haar property there. As mentioned above U contains no two-dimensional Haar subspace. Hence there exists a nontrivial function $w = c_1 v_1 + c_2 v_2$ with at most one zero in $(-1, 1)$ and at least two zeros in $[-1, 1]$. Then the function $w + dv_1$ where d is a sufficiently small real number has at least two zeros in $(-1, 1)$, a contradiction.

Now we study the class of those spaces which contain normed sign-regular Markoff systems. To do this let M and $\tilde{M}$ be defined as in Theorem 2.2. Moreover, let U denote an n-dimensional subspace of $F(M)$ which contains a normed sign-regular Markoff system on M. Then it follows from [3, Theorem 3]:

There exist a basis $\{g_1, ..., g_n\}$ of U, a strictly increasing function $h \in F(M)$, continuous strictly increasing functions $w_1, ..., w_{n-1}$ defined on the interval $J = (\inf h(M), \sup h(M))$, and $c \in J$ such that for every $x \in M$,

$$\begin{aligned} g_1(x) &= 1 \\ g_2(x) &= \int_c^{h(x)} dw_1(t_1) \\ &\vdots \\ g_n(x) &= \int_c^{h(x)} \int_c^{t_1} \cdots \int_c^{t_{n-2}} dw_{n-1}(t_{n-1}) \cdots dw_1(t_1). \end{aligned} \tag{2.13}$$

THEOREM 2.6. *The following conditions are equivalent*:

There exists an n-dimensional subspace $\tilde{U}$ *of* $F(M)$ *such that* $1 \in \tilde{U}$, $\tilde{U}$ *contains a sign-regular Descartes system on* $\tilde{M}$, *and* $\tilde{U}|_M = U$; (2.14)

there exists a representation of the type (2.13) *such that* $\inf h(M) \in \mathbb{R}$ *and* $\lim w_i(x)$ *exists as* x *tends to* $\inf h(M)$, $1 \leqslant i \leqslant n-1$. (2.15)

Proof. We first show that (2.15) implies (2.14). Let $d < \inf M$ and $\tilde{d} < \inf h(M)$ be some real numbers. We extend the functions h and $w_1, ..., w_{n-1}$ as follows: Let $\tilde{h}(x) = h(x)$ for every $x \in M$ and let $\tilde{h}$ be strictly increasing on $(d, \inf M] \cup M$. Let $\tilde{w}_i(x) = w_i(x)$ for every $x \in J$ and let $\tilde{w}_i$ be strictly increasing and continuous on $(\tilde{d}, \inf h(M)] \cup J$. Now we define for $c \in J$ and every $x \in (d, \inf M] \cup M$,

$$\tilde{g}_1(x) = 1$$

$$\tilde{g}_2(x) = \int_c^{\tilde{h}(x)} d\tilde{w}_1(t_1)$$

$$\vdots$$

$$\tilde{g}_n(x) = \int_c^{\tilde{h}(x)} \int_c^{t_1} \cdots \int_c^{t_{n-2}} d\tilde{w}_{n-1}(t_{n-1}) \cdots d\tilde{w}_1(t_1).$$

Then it follows from [1] that span $\{\tilde{g}_1, ..., \tilde{g}_n\}$ is a sign-regular Markoff system on $(d, \inf M] \cup M$. Moreover it is obvious that $\tilde{g}_i(x) = g_i(x)$ for every $x \in M$, $1 \leqslant i \leqslant n$.

Set $\tilde{U} = \text{span}\,\{\tilde{g}_1|_{\tilde{M}}, ..., \tilde{g}_n|_{\tilde{M}}\}$. Then the statement (2.14) follows directly from Theorem 2.1.

Now we show that (2.14) implies (2.15). Let $d > 0$ be given. Then by Theorem 2.2 there exists a space $\tilde{U}_d$ on $\hat{M} = (\inf M - d, \inf M) \cup \tilde{M}$ such that $\tilde{U}_d|_{\tilde{M}} = \tilde{U}$ and $\tilde{U}_d$ contains a normed sign-regular Markoff system on $\hat{M}$. By [3, Theorem 3] there exists a representation of the type (2.13) for $\tilde{U}_d$ on $\hat{M}$.

This implies the statements in (2.15).

Finally we give a result concerning the extension of normed sign-regular Markoff systems.

PROPOSITION 2.7. *Let* $\inf M \notin M$, $\inf M \in \mathbb{R}$. *Assume that for any points* $x, y \in M$ *with* $x < y$ *there exists a point* $z \in M$ *with* $x < z < y$. *Set* $\tilde{M} = \{\inf M\} \cup M$. *Let* U *denote an n-dimensional subspace of* $F(\tilde{M})$ *such that* $u(\inf M) = \lim_{x \to \inf M, x \in M} u(x)$ *for every* $u \in U$. *Moreover assume that* U *contains a normed sign-regular Markoff system* $\{u_1, ..., u_n\}$ *on* M. *Then this system is even a normed sign-regular Markoff system on* $\tilde{M}$.

Proof. Let the system $\{u_1, ..., u_i\}$ be given. Assume that there exists a function $\tilde{u} \in \text{span}\,\{u_1, ..., u_i\} \setminus \{0\}$ and points $x_1 < \cdots < x_{i+1}$ in $\tilde{M}$ with $(-1)^j \tilde{u}(x_j) \geqslant 0$, $1 \leqslant j \leqslant i+1$. By hypothesis, $x_1 = \inf M$. If there exists a point $z \in (x_1, x_2) \cap M$ with $\tilde{u}(z) \leqslant 0$, then setting $y_1 = z$, $y_j = x_j$, $2 \leqslant j \leqslant i+1$, we have

$$(-1)^j \tilde{u}(y_j) \geqslant 0, \qquad 1 \leqslant j \leqslant i+1,$$

a contradiction to Theorem 8.8 in [2].

Therefore assume that $\tilde{u}(x) > 0$ for every $x \in (x_1, x_2) \cap M$. Then it follows from $\tilde{u}(\inf M) = \lim_{x \to \inf M, x \in M} \tilde{u}(x)$ that there exist two points $y < z$ in $(x_1, x_2) \cap M$ with $\tilde{u}(y) < \tilde{u}(z)$. Then setting $y_1 = y, y_2 = z$, and $y_j = x_j$, $3 \leqslant j \leqslant i+1$, we have

$$(-1)^j \left(\tilde{u}(y_{j+1}) - \tilde{u}(y_j)\right) \leqslant 0, \qquad 1 \leqslant j \leqslant i,$$

a contradiction.

COROLLARY 2.8. *Let $M = (a, b)$ (resp. $M = (a, b]$ and $M = [a, b)$) be a real subinterval, and let U denote an n-dimensional subspace of $C[a, b]$ such that U has a normed Markoff system on (a, b). Then U has a normed Markoff system on $[a, b]$.*

This result follows directly from Proposition 2.7.

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